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Some Cauchy mean-type mappings for which the geometric mean is invariant

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ABSTRACT

The invariance of the geometric mean G with respect to the Cauchy mean-type mapping $(D^{f,g}, D^{h,k})$, i.e. the equation $G \circ (D^{f,g}, D^{h,k}) = G$, is considered. We give some necessary, and necessary and sufficient conditions under assumption that one of the generators of each Cauchy means is a power function.

1. Introduction

By $D^{f,g}$ denote the Cauchy mean generated by the functions f and g on an interval $I \subset \mathbb{R}$ and by G, the geometric mean. The problem of invariance of the geometric mean G in the class of the Cauchy means reduces to the functional equation

$$G \circ \left(D^{f,g}, D^{h,k}\right) = G. \tag{1.1}$$

In this paper we consider this functional equation under assumption that one of the generators of each Cauchy's means is a power function, i.e.

$$g(x) = x^p, \qquad k(x) = x^r, \quad x \in I, \tag{1.2}$$

for some $p, r \in \mathbb{R}\backslash\{0\}$.

One of the consequences of the invariance is the convergence of the sequence of iterates of mapping $(D^{f,g}, D^{h,k})$ satisfying this equation to the mean-type mapping (G, G) (cf. [1,2]).

In recent years the problem of invariance was the subject of research of many authors. The invariance of the arithmetic mean with respect to two quasi-arithmetic means was first investigated by Matkowski [3] under twice continuous differentiability assumptions. Finally this problem was completely solved by Daróczy and Páles [4], assuming only continuity of the unknown functions. The invariance of the arithmetic mean with respect to Lagrangian means was the subject of investigation of paper [5] by Matkowski. The invariance of the arithmetic, geometric and harmonic means with respect to the so-called Beckenbach–Gini means was studied by Matkowski in [6].

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Let us mention that the problem of invariance of the geometric mean in the class of the Lagrangian means has been solved in [7] and [8]. All pairs (M, N) of Stolarsky's means for which the geometric mean is invariant have been determined in [9].

Section 2 is devoted to some basic definitions, auxiliary results and a motivation. In Section 3, assuming that the functions g and k are given by (1.2), we prove that the unknown functions f and h satisfying Eq. (1.1) must be of high class regularity. Applying this fact, in Section 4, we give some necessary conditions which reduces the problem to a differential equation. In Section 5, assuming that $r = -p$, we prove that the functions f, g, h, k satisfy Eq. (1.1) if, and only if, either there exists $q \in \mathbb{R}\backslash\{0, p\}$ such that for all $x \in I$

$$f(x) = A_1x^q + B_1x^p + C_1, \qquad h(x) = A_2x^{-q} + B_2x^{-p} + C_2,$$

or there exists $q \in \{0, p\}$ such that for all $x \in I$

$$f(x) = A_1x^q \ln x + B_1x^p + C_1, \qquad h(x) = A_2x^{-q} \ln x + B_2x^{-p} + C_2,$$

for some $A_i, B_i, C_i \in \mathbb{R}$, $A_i \neq 0$, $i \in \{1, 2\}$.

It follows that the geometric mean G is $(D^{f,g}, D^{h,k})$-invariant if, and only if, there exists $q \in \mathbb{R}$ such that

$$D^{f,g} = E_{p,q} \quad \text{and} \quad D^{h,k} = E_{-p,-q},$$

where $E_{p,q}$ stands for Stolarsky's mean.

2. Definitions and motivation

Let $I \subset \mathbb{R}$ be an interval. A function $M : I^2 \to \mathbb{R}$ is said to be a *mean* on I if

$$\min(x, y) \leqslant M(x, y) \leqslant \max(x, y), \quad x, y \in I.$$

If moreover for all $x, y \in I$, $x \neq y$, these inequalities are sharp, the mean M is called *strict*, and M is called *symmetric*, if for all $x, y \in I$, $M(x, y) = M(y, x)$.

Note that if $M : I^2 \to \mathbb{R}$ is a mean, then M is *reflexive*, that is,

$$M(x, x) = x, \quad x \in I,$$

and, consequently, for every interval $J \subset I$ we have $M(J^2) = J$; in particular, $M(I^2) = I$. (For more information about means cf., for instance, [1,10].)

Let $M : I^2 \to I$, $N : I^2 \to I$ be means. Following Matkowski [2], we call a mean $K : I^2 \to I$ *invariant with respect to the mean-type mapping* $(M, N) : I^2 \to I^2$, shortly, (M, N)*-invariant*, if

$$K\big(M(x, y), N(x, y)\big) = K(x, y), \quad x, y \in I.$$

As a motivation for this paper let us quote the following

Proposition 1. *(Cf. [2].) Let $I \subset \mathbb{R}$ be an interval. If $(M, N) : I^2 \to I^2$ is a continuous mean-type mapping such that*

$$0 < \max\big(M(x, y), N(x, y)\big) - \min\big(M(x, y), N(x, y)\big) < \max(x, y) - \min(x, y),$$

then:

1. *there is a continuous mean $K : I^2 \to I$ such that the sequence of iterates $((M, N)^n)_{n=1}^{\infty}$ of the mapping (M, N) converges (pointwise) to a continuous mean-type mapping $(K, K) : I^2 \to I^2$*;
2. *K is (M, N)-invariant*;
3. *a continuous (M, N)-invariant mean-type mapping is unique*;
4. *if M and N are strict means then so is K.*

Remark 1. This proposition improves a well-known result in which it is assumed that both means M and N are strict (cf. for instance [1]). A unique continuous (M, N)-invariant mean K is also called the Gauss composition of M and N. Moreover the sequence of iterates of the mean-type mapping $(M, N) : I^2 \to I^2$ is called the Gauss-iteration (cf. [1]).

Let us remind the definition of a Cauchy mean (cf. [11]).

Let $I \subset \mathbb{R}$ be an interval. Let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be differentiable functions such that $g'(x) \neq 0$, for all $x \in I$, and the function $\frac{f'}{g'}$ be injective. Then, by the Cauchy mean value theorem, the function $D^{f,g} : I^2 \to I$ given by

$$D^{f,g}(x, y) = \begin{cases} (\frac{f'}{g'})^{-1}(\frac{f(x)-f(y)}{g(x)-g(y)}), & x \neq y, \\ x, & x = y \end{cases}$$

is correctly defined and it is called *a Cauchy mean* generated by the functions f and g.

Let us note the following

Remark 2. (Cf. [11].) Let $I \subset \mathbb{R}$ be an interval and suppose that $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are differentiable functions and $g'(x) \neq 0$, for all $x \in I$. If the function $\frac{f'}{g'}$ is one-to-one, then it is strictly monotonic and continuous.

3. A regularity theorem

Throughout this paper we assume that $I \subset (0, +\infty)$ is an open interval.

Theorem 1. *Let the functions $g : I \to \mathbb{R}$ and $k : I \to \mathbb{R}$ be given by*

$$g(x) = x^p, \qquad k(x) = x^r, \quad x \in I,$$

for some $p, r \in \mathbb{R}\backslash\{0\}$. Suppose that $f : I \to \mathbb{R}$ and $h : I \to \mathbb{R}$ are differentiable functions such that $\frac{f'}{g'}$ and $\frac{h'}{k'}$ are injective in I.

If the geometric mean G is $(D^{f,g}, D^{h,k})$-invariant, i.e. for all $x, y \in I$, $x \neq y$,

$$\left(\frac{f'}{g'}\right)^{-1}\left(\frac{f(x)-f(y)}{g(x)-g(y)}\right)\cdot\left(\frac{h'}{k'}\right)^{-1}\left(\frac{h(x)-h(y)}{k(x)-k(y)}\right) = x \cdot y, \tag{3.1}$$

then for every $x_0 \in I$ and every $n \in \mathbb{N}$ there exists a neighbourhood U of x_0 such that f and h are of the class C^n in U except for a closed set with an empty interior.

Proof. Observe that, by Remark 2, the functions $\frac{f'}{g'}$ and $\frac{h'}{k'}$ are continuous in I. Hence we obtain that the functions f and h are continuously differentiable in I.

Assume first that for every $x_0 \in I$ there exists a $y_0 \in I$, $x_0 \neq y_0$, such that

$$\frac{h(x_0)-h(y_0)}{x_0^r - y_0^r} \neq \frac{x_0 h'(x_0) + y_0 h'(y_0)}{r(x_0^r + y_0^r)}.$$

Let us fix an $x_0 \in I$, put

$$u_0 := \frac{f(x_0)-f(y_0)}{x_0^p - y_0^p}, \qquad \Delta := \{(x,x)\colon x \in I\},$$

and define the function $\Phi : (I^2\backslash\Delta) \times \mathbb{R} \to \mathbb{R}$ by

$$\Phi(x, y, u) := \frac{f(x)-f(y)}{x_0^p - y_0^p} - u.$$

Note that Φ is of the class C^1, $\Phi(x_0, y_0, u_0) = 0$, and

$$\frac{\partial \Phi}{\partial y}(x_0, y_0, u_0) = \frac{f'(y_0)(y_0^p - x_0^p) - p y_0^{p-1}(f(y_0) - f(x_0))}{(x_0^p - y_0^p)^2} \neq 0.$$

Indeed, if the last relation were not true, we would have

$$\frac{f(x_0)-f(y_0)}{x_0^p - y_0^p} = \frac{f'(y_0)}{p y_0^{p-1}},$$

and, by the Cauchy mean value theorem,

$$\frac{f(x_0)-f(y_0)}{x_0^p - y_0^p} = \frac{f'(\xi)}{p \xi^{p-1}},$$

for some $\xi \neq y_0$, whence

$$\frac{f'(y_0)}{p y_0^{p-1}} = \frac{f'(\xi)}{p \xi^{p-1}}.$$

This is a contradiction as $\frac{f'}{g'}$ is one-to-one.

By the implicit function theorem, there exist a neighbourhood D of the point (x_0, u_0),

$$D = (x_0 - \delta, x_0 + \delta) \times (u_0 - \delta, u_0 + \delta)$$

for some $\delta > 0$, and a unique function $\alpha : D \to I$ of the class C^1 in D such that

$$\alpha(x_0, u_0) = y_0, \qquad \Phi\big(x, \alpha(x,u), u\big) = 0, \quad (x,u) \in D,$$

that is

$$\alpha(x_0, u_0) = y_0, \qquad \frac{f(x) - f(\alpha(x,u))}{g(x) - g(\alpha(x,u))} = \frac{f(x) - f(\alpha(x,u))}{x^p - (\alpha(x,u))^p} = u, \quad (x,u) \in D.$$

Setting $y = \alpha(x,u)$ in (3.1), we obtain

$$\left(\frac{f'}{g'}\right)^{-1}(u) \cdot \left(\frac{h'}{k'}\right)^{-1}\left(\frac{h(x) - h(\alpha(x,u))}{k(x) - k(\alpha(x,u))}\right) = x \cdot \alpha(x,u), \quad (x,u) \in D. \tag{3.2}$$

Put

$$v_0 := \frac{h(x_0) - h(\alpha(x_0,u_0))}{k(x_0) - k(\alpha(x_0,u_0))} = \frac{h(x_0) - h(y_0)}{x_0^r - y_0^r},$$

and define the function $\Psi : D \times \mathbb{R} \to \mathbb{R}$ by

$$\Psi(x,u,v) := \frac{h(x) - h(\alpha(x,u))}{k(x) - k(\alpha(x,u))} - v = \frac{h(x) - h(\alpha(x,u))}{x^r - (\alpha(x,u))^r} - v, \quad (x,u) \in D,\ v \in \mathbb{R}.$$

Note that Ψ is of the class C^1, $\Psi(x_0,u_0,v_0) = 0$, and

$$\begin{aligned}\frac{\partial \Psi}{\partial x}(x_0,u_0,v_0) = {} & \frac{[h'(x_0) - h'(\alpha(x_0,u_0))\frac{\partial\alpha}{\partial x}(x_0,u_0)](x_0^r - (\alpha(x_0,u_0))^r)}{(x_0^r - (\alpha(x_0,u_0))^r)^2} \\ & - \frac{[rx_0^{r-1} - r(\alpha(x_0,u_0))^{r-1}\frac{\partial\alpha}{\partial x}(x_0,u_0)](h(x_0) - h(\alpha(x_0,u_0)))}{(x_0^r - (\alpha(x_0,u_0))^r)^2}.\end{aligned}$$

Suppose first that

$$\frac{\partial \Psi}{\partial x}(x_0,u_0,v_0) \neq 0.$$

Then, by the implicit function theorem there exist a neighbourhood W of the point (u_0, v_0),

$$W = (u_0 - \rho, u_0 + \rho) \times (v_0 - \rho, v_0 + \rho),$$

for some $\rho > 0$, and a unique function $\beta : W \to I$ of the class C^1 in W such that

$$\beta(u_0,v_0) = x_0, \qquad \Psi\big(\beta(u,v), u, v\big) = 0, \quad (u,v) \in W,$$

that is

$$\beta(u_0,v_0) = x_0, \qquad \frac{h(\beta(u,v)) - h(\alpha(\beta(u,v),u))}{k(\beta(u,v)) - k(\alpha(\beta(u,v),u))} = v, \quad (u,v) \in W.$$

Substituting $x = \beta(u,v)$ in (3.2), we obtain

$$\left(\frac{f'}{g'}\right)^{-1}(u) \cdot \left(\frac{h'}{k'}\right)^{-1}(v) = \beta(u,v) \cdot \alpha\big(\beta(u,v), u\big), \quad (u,v) \in W.$$

Since the right-hand side is a function of the class C^1 in W, we infer that $(\frac{f'}{g'})^{-1}$ and $(\frac{h'}{k'})^{-1}$ are of the class C^1 in the intervals $(u_0 - \rho, u_0 + \rho)$ and $(v_0 - \rho, v_0 + \rho)$, respectively. Since the sets

$$\left\{u \colon \left(\left(\frac{f'}{g'}\right)^{-1}\right)'(u) = 0\right\}, \qquad \left\{v \colon \left(\left(\frac{h'}{k'}\right)^{-1}\right)'(v) = 0\right\}$$

are nowhere dense, it follows that the functions $\frac{f'}{g'}$ and $\frac{h'}{k'}$ are of the class C^1 in an open nonempty subinterval contained in $(x_0 - \delta, x_0 + \delta)$, and consequently the functions f and h are of the class C^2 in that subinterval.

Suppose now that

$$\frac{\partial \Psi}{\partial x}(x_0,u_0,v_0) = 0.$$

Then, by the definition of Ψ,

$$\frac{\partial \Psi}{\partial x}(x_0, u_0, v) = 0, \quad v \in \mathbb{R}.$$

If there exists a point $(x_1, u_1) \in D$ such that

$$\frac{\partial \Psi}{\partial x}(x_1, u_1, v) \neq 0,$$

then we can choose a $\delta_1 > 0$ such that $D_1 := (x_1 - \delta_1, x_1 + \delta_1) \times (u_1 - \delta_1, u_1 + \delta_1) \subset D$, and

$$\frac{\partial \beta}{\partial x}(x, u) \neq 0, \quad (x, u) \in D_1,$$

and we can repeat the above reasoning with (x_0, u_0) and D replaced by (x_1, u_1) and D_1, respectively.

If there were no a point $(x_1, u_1) \in D$ such that

$$\frac{\partial \Psi}{\partial x}(x_1, u_1, v) \neq 0,$$

then

$$\frac{\partial \Psi}{\partial x}(x, u, v) = 0, \quad (x, u) \in D, \ v \in \mathbb{R}.$$

Hence, by the definition of Ψ,

$$\left[h'(x) - h'\big(\alpha(x, u)\big)\frac{\partial \alpha}{\partial x}(x, u)\right]\left[x^r - \big(\alpha(x, u)\big)^r\right] = \left[rx^{r-1} - r\big(\alpha(x, u)\big)^{r-1}\frac{\partial \alpha}{\partial x}(x, u)\right]\left[h(x) - h\big(\alpha(x, u)\big)\right],$$

for all $(x, u) \in D$. As in this case the function on the right-hand side of (3.1) does not depend on x,

$$\frac{\partial \alpha}{\partial x}(x, u) = -\frac{\alpha(x, u)}{x}, \quad (x, u) \in D,$$

whence, after simple calculations,

$$\left[xh'(x) + \alpha(x, u)h'\big(\alpha(x, u)\big)\right]\left[x^r - \big(\alpha(x, u)\big)^r\right] = \left[rx^r + r\big(\alpha(x, u)\big)^r\right]\left[h(x) - h\big(\alpha(x, u)\big)\right].$$

Consequently, setting $y := \alpha(x, u)$, we would get

$$\left[xh'(x) + yh'(y)\right]\left[x^r - y^r\right] = \left[rx^r + ry^r\right]\left[h(x) - h(y)\right],$$

for all $x \in (x_0 - \delta, x_0 + \delta)$, $y \in (y_0 - \varepsilon, y_0 + \varepsilon)$, for some $\varepsilon > 0$. In particular,

$$\frac{h(x_0) - h(y_0)}{x_0^r - y_0^r} = \frac{x_0 h'(x_0) + y_0 h'(y_0)}{r(x_0^r + y_0^r)},$$

which contradicts to the assumption.

To finish the proof assume that there exists an $x_0 \in I$ such that for all $y \in I$, $y \neq x_0$,

$$\frac{h(x_0) - h(y_0)}{x_0^r - y_0^r} = \frac{x_0 h'(x_0) + y_0 h'(y_0)}{r(x_0^r + y_0^r)}.$$

Then

$$h'(y) = \frac{h(x_0) - h(y)}{x_0^r - y^r} \cdot \frac{r(x_0^r + y_0^r)}{y} - \frac{x_0 h'(x_0)}{y}, \quad y \in I \backslash \{x_0\},$$

which implies that h is of the class C^2 in $I \backslash \{x_0\}$. From (3.1) we infer that so is f.

Now, an obvious induction proves that for every $n \in \mathbb{N}$ the functions f and h are of the class C^n in an open nonempty subinterval contained in $(x_0 - \delta, x_0 + \delta)$. $\square$

Let us mention that Theorem 1 is a counterpart of a result by J. Matkowski [5], where the arithmetic mean was considered.

4. Some necessary conditions for $(D^{f,g}, D^{h,k})$-invariance of the geometric mean

Lemma 1. *Let the functions $g : I \to \mathbb{R}$ and $k : I \to \mathbb{R}$ be given by*

$$g(x) = x^p, \qquad k(x) = x^r, \quad x \in I,$$

for some $p, r \in \mathbb{R} \setminus \{0\}$. Suppose that $f : I \to \mathbb{R}$ and $h : I \to \mathbb{R}$ are differentiable functions such that $\frac{f'}{g'}$ and $\frac{h'}{k'}$ are injective in I.

If the functions f, g, h, k satisfy Eq. (3.1), then for every nonempty open subinterval $J \subset I$ there exist a nonempty open subinterval $I_0 \subset J$ and $c \neq 0$, $c = c(I_0)$, such that functions f and h are of the class C^3 in I_0 and

$$\left(\frac{f'(x)}{px^{p-1}}\right)' \cdot \left(\frac{h'(x)}{rx^{r-1}}\right)' = \frac{c}{x^{2(p+r+1)}}, \quad x \in I_0.$$

Proof. Assume that the functions f, g, h, k satisfy Eq. (3.1) and take an arbitrary open interval $J \subset I$. Theorem 1 and the injectivity of functions $\frac{f'}{g'}$ and $\frac{h'}{k'}$ imply that there exists a maximal, nonempty open subinterval $I_0 \subset J$ such that f and g are of the class C^3 in I_0, and

$$\left(\frac{f'(x)}{px^{p-1}}\right)' \cdot \left(\frac{h'(x)}{rx^{r-1}}\right)' \neq 0, \quad x \in I_0.$$

Define the functions $u : I_0 \to \mathbb{R}$, $v : I_0 \to \mathbb{R}$, $B_f : I_0 \times I_0 \to \mathbb{R}$ and $B_h : I_0 \times I_0 \to \mathbb{R}$ as follows

$$u(x) := \frac{f'(x)}{px^{p-1}}, \qquad v(x) := \frac{h'(x)}{rx^{r-1}}, \tag{4.1}$$

$$B_f(x, y) := \begin{cases} \frac{f(x)-f(y)}{x^p-y^p}, & x \neq y, \\ u(x), & x = y, \end{cases} \qquad B_h(x, y) := \begin{cases} \frac{h(x)-h(y)}{x^r-y^r}, & x \neq y, \\ v(x), & x = y. \end{cases} \tag{4.2}$$

By the assumptions the functions u, v, B_f and B_h are of the class C^2 in I_0 and $I_0 \times I_0$, respectively. Moreover

$$u'(x) \cdot v'(x) \neq 0, \quad x \in I_0.$$

Now we can write Eq. (3.1) in the form

$$u^{-1}\big(B_f(x, y)\big) \cdot v^{-1}\big(B_h(x, y)\big) = x \cdot y, \quad x, y \in I_0,\ x \neq y. \tag{4.3}$$

Differentiating twice with respect to x both sides of this equation, we get

$$\begin{aligned}
&\left(\frac{1}{u'(u^{-1}(B_f(x, y)))}\frac{\partial^2 B_f}{\partial x^2}(x, y) - \frac{u''(u^{-1}(B_f(x, y)))}{(u'(u^{-1}(B_f(x, y))))^3}\left(\frac{\partial B_f}{\partial x}(x, y)\right)^2\right)v^{-1}\big(B_h(x, y)\big) \\
&\quad + \left(\frac{1}{v'(v^{-1}(B_h(x, y)))}\frac{\partial^2 B_h}{\partial x^2}(x, y) - \frac{v''(v^{-1}(B_h(x, y)))}{(v'(v^{-1}(B_h(x, y))))^3}\left(\frac{\partial B_h}{\partial x}(x, y)\right)^2\right)u^{-1}\big(B_f(x, y)\big) \\
&\quad + \frac{2}{u'(u^{-1}(B_f(x, y)))v'(v^{-1}(B_h(x, y)))}\frac{\partial B_f}{\partial x}(x, y)\frac{\partial B_h}{\partial x}(x, y) = 0,
\end{aligned} \tag{4.4}$$

where

$$\frac{\partial B_f}{\partial x}(x, y) = \frac{(x^p - y^p)f'(x) - px^{p-1}(f(x) - f(y))}{(x^p - y^p)^2}, \tag{4.5}$$

$$\frac{\partial^2 B_f}{\partial x^2}(x, y) = \frac{(x^p - y^p)^2 f''(x) - 2px^{p-1}(x^p - y^p)f'(x) + px^{p-2}((p+1)x^p + (p-1)y^p)(f(x) - f(y))}{(x^p - y^p)^3}. \tag{4.6}$$

Replacing f by h and p by r in (4.5) and (4.6) we obtain, respectively, the formulas for $\frac{\partial B_h}{\partial x}(x, y)$ and $\frac{\partial^2 B_h}{\partial x^2}(x, y)$, for all $x, y \in I_0$, $x \neq y$.

By de l'Hospital's rule it is easy to show that

$$\lim_{y \to x} \frac{\partial B_f}{\partial x}(x, y) = \frac{xf''(x) - (p-1)f'(x)}{2px^p}, \tag{4.7}$$

$$\lim_{y \to x} \frac{\partial^2 B_f}{\partial x^2}(x, y) = \frac{2x^2 f^{(3)}(x) - 3(p-1)xf''(x) + (p^2-1)f'(x)}{6px^{p+1}}. \tag{4.8}$$

Obviously, if we replace f by h and p by r in (4.7) and (4.8) we get, respectively, $\lim_{y \to x} \frac{\partial B_h}{\partial x}(x, y)$ and $\lim_{y \to x} \frac{\partial^2 B_h}{\partial x^2}(x, y)$.

Letting $y \to x$ in (4.4), by (4.7), (4.8) and (4.1), after simple calculations we obtain, for all $x \in I_0$,

$$\frac{x^2 f^{(3)}(x) - (p-1)(p-2)f'(x)}{xf''(x) - (p-1)f'(x)} + \frac{x^2 h^{(3)}(x) - (r-1)(r-2)h'(x)}{xh''(x) - (r-1)h'(x)} = -6. \tag{4.9}$$

Now notice that, for all $x \in I_0$, we have

$$x^2 f^{(3)}(x) - (p-1)(p-2)f'(x) = x\big(xf''(x) - (p-1)f'(x)\big)' + (p-2)\big(xf''(x) - (p-1)f'(x)\big),$$

and similarly for the function h with the parameter r. Hence, by Eq. (4.9), we get

$$\frac{(xf''(x) - (p-1)f'(x))'}{xf''(x) - (p-1)f'(x)} + \frac{(xh''(x) - (r-1)h'(x))'}{xh''(x) - (r-1)h'(x)} = -\frac{2+p+r}{x}, \quad x \in I_0,$$

whence, for some $c_1 \in \mathbb{R}\backslash\{0\}$,

$$\big(xf''(x) - (p-1)f'(x)\big)\big(xh''(x) - (r-1)h'(x)\big) = \frac{c_1}{x^{2+p+r}}, \quad x \in I_0. \tag{4.10}$$

Since

$$xf''(x) - (p-1)f'(x) = px^p\left(\frac{f'(x)}{px^{p-1}}\right)', \quad x \in I_0,$$

and

$$xh''(x) - (r-1)h'(x) = rx^r\left(\frac{h'(x)}{rx^{r-1}}\right)', \quad x \in I_0,$$

equality (4.10) gives

$$\left(\frac{f'(x)}{px^{p-1}}\right)'\left(\frac{h'(x)}{rx^{r-1}}\right)' = \frac{c}{x^{2(1+p+r)}}, \quad x \in I_0,$$

where $c := \frac{c_1}{pr}$ and $c \neq 0$. $\square$

Applying this result we prove the following

Lemma 2. *Let the functions $g : I \to \mathbb{R}$ and $k : I \to \mathbb{R}$ be given by*

$$g(x) = x^p, \qquad k(x) = x^r, \quad x \in I,$$

for some $p, r \in \mathbb{R}\backslash\{0\}$. Suppose that $f : I \to \mathbb{R}$ and $h : I \to \mathbb{R}$ are differentiable functions such that $\frac{f'}{g'}$ and $\frac{h'}{k'}$ are injective in I.

If the functions f, g, h, k satisfy Eq. (3.1), then for every nonempty open subinterval $J \subset I$ there exists a nonempty open subinterval $I_0 \subset J$ such that f and h are of the class C^5 in I_0 and:

(i) *the function $w_f : I_0 \to \mathbb{R}$ defined by*

$$w_f(x) := \frac{x^2 f^{(3)}(x) - 2(p-1)xf''(x) + p(p-1)f'(x)}{x(xf''(x) - (p-1)f'(x))}, \quad x \in I_0, \tag{4.11}$$

satisfies in the interval I_0 the equation

$$\begin{aligned} &2(p+r)(p+r+1)(2p+1) + 9x^3 w_f(x)w_f'(x) + \big(2(p+r)+9\big)x^2\big(w_f(x)\big)^2 \\ &\quad + 3(5p+r+3)x^2 w_f'(x) + \big[(p+r)\big(2(p+r)+4p+7\big) + 12p + 9\big]xw_f(x) = 0; \end{aligned} \tag{4.12}$$

(ii) *the function $w_h : I_0 \to \mathbb{R}$ defined by*

$$w_h(x) := \frac{x^2 h^{(3)}(x) - 2(r-1)xh''(x) + r(r-1)h'(x)}{x(xh''(x) - (r-1)h'(x))}, \quad x \in I_0,$$

satisfies in the interval I_0 the equation

$$\begin{aligned} &2(p+r)(p+r+1)(2r+1) + 9x^3 w_h(x)w_h'(x) + \big(2(p+r)+9\big)x^2\big(w_h(x)\big)^2 \\ &\quad + 3(5r+p+3)x^2 w_h'(x) + \big[(p+r)\big(2(p+r)+4r+7\big) + 12r + 9\big]xw_h(x) = 0. \end{aligned}$$

Proof. Assume that the functions f, g, h, k satisfy Eq. (3.1) and take an arbitrary open interval $J \subset I$. By Theorem 1 and injectivity of the functions $\frac{f'}{g'}$ and $\frac{h'}{k'}$ there exists a nonempty open and maximal (in the sense of inclusion) subinterval $I_0 \subset J$ such that f and g are of the class C^5 in it and

$$\left(\frac{f'(x)}{px^{p-1}}\right)' \cdot \left(\frac{h'(x)}{rx^{r-1}}\right)' \neq 0, \quad x \in I_0.$$

The functions u, v, B_f and B_h defined by (4.1) and (4.2) (see the proof of Lemma 1) are of the class C^4 in I_0 and $I_0 \times I_0$, respectively.

Differentiating four times with respect to x both sides of Eq. (4.3), for all $x, y \in I_0$, we get

$$\begin{aligned}
&\left[\frac{1}{u'}\frac{\partial^4 B_f}{\partial x^4} - \left(3\left(\frac{\partial^2 B_f}{\partial x^2}\right)^2 + 4\frac{\partial B_f}{\partial x}\frac{\partial^3 B_f}{\partial x^3}\right)\frac{u''}{(u')^3} + 6\left(\frac{\partial B_f}{\partial x}\right)^2\frac{\partial^2 B_f}{\partial x^2}\left(3\frac{(u'')^2}{(u')^5} - \frac{u^{(3)}}{(u')^4}\right)\right.\\
&\quad\left. + \left(\frac{\partial B_f}{\partial x}\right)^4\left(10\frac{u''u^{(3)}}{(u')^6} - 15\frac{(u'')^3}{(u')^7} - \frac{u^{(4)}}{(u')^5}\right)\right]v^{-1}(B_h)\\
&\quad + \frac{4}{v'}\frac{\partial B_h}{\partial x}\left[\frac{1}{u'}\frac{\partial^3 B_f}{\partial x^3} - 3\frac{u''}{(u')^3}\frac{\partial B_f}{\partial x}\frac{\partial^2 B_f}{\partial x^2} + \left(\frac{\partial B_f}{\partial x}\right)^3\left(3\frac{(u'')^2}{(u')^5} - \frac{u^{(3)}}{(u')^4}\right)\right]\\
&\quad + 6\left(\frac{1}{u'}\frac{\partial^2 B_f}{\partial x^2} - \frac{u''}{(u')^3}\left(\frac{\partial B_f}{\partial x}\right)^2\right)\left(\frac{1}{v'}\frac{\partial^2 B_h}{\partial x^2} - \frac{v''}{(v')^3}\left(\frac{\partial B_h}{\partial x}\right)^2\right)\\
&\quad + \frac{4}{u'}\frac{\partial B_f}{\partial x}\left[\frac{1}{v'}\frac{\partial^3 B_h}{\partial x^3} - 3\frac{v''}{(v')^3}\frac{\partial B_h}{\partial x}\frac{\partial^2 B_h}{\partial x^2} + \left(\frac{\partial B_h}{\partial x}\right)^3\left(3\frac{(v'')^2}{(v')^5} - \frac{v^{(3)}}{(v')^4}\right)\right]\\
&\quad + \left[\frac{1}{v'}\frac{\partial^4 B_h}{\partial x^4} - \left(3\left(\frac{\partial^2 B_h}{\partial x^2}\right)^2 + 4\frac{\partial B_h}{\partial x}\frac{\partial^3 B_h}{\partial x^3}\right)\frac{v''}{(v')^3} + 6\left(\frac{\partial B_h}{\partial x}\right)^2\frac{\partial^2 B_h}{\partial x^2}\left(3\frac{(v'')^2}{(v')^5} - \frac{v^{(3)}}{(v')^4}\right)\right.\\
&\quad\left. + \left(\frac{\partial B_h}{\partial x}\right)^4\left(10\frac{v''v^{(3)}}{(v')^6} - 15\frac{(v'')^3}{(v')^7} - \frac{v^{(4)}}{(v')^5}\right)\right]u^{-1}(B_f) = 0,
\end{aligned}\tag{4.13}$$

where

$$u^{(i)}, \qquad v^{(i)}, \qquad \frac{\partial^i B_f}{\partial x^i}, \qquad \frac{\partial^i B_h}{\partial x^i},$$

stand, respectively, for

$$u^{(i)}\big(u^{-1}\big(B_f(x,y)\big)\big), \qquad v^{(i)}\big(v^{-1}\big(B_h(x,y)\big)\big), \qquad \frac{\partial^i B_f}{\partial x^i}(x,y), \qquad \frac{\partial^i B_h}{\partial x^i}(x,y),$$

for all $i \in \{1,2,3,4\}$ and $x, y \in I_0$. By the definition of B_f (see (4.2)) we get the form for $\frac{\partial^i B_f}{\partial x^i}$ for all $i \in \{1,2,3,4\}$. Namely, $\frac{\partial B_f}{\partial x}$ and $\frac{\partial^2 B_f}{\partial x^2}$ are given by (4.5) and (4.6), and

$$\begin{aligned}
\frac{\partial^3 B_f}{\partial x^3}(x,y) &= \frac{f^{(3)}(x)}{x^p - y^p} - \frac{3px^{p-1}f''(x)}{(x^p - y^p)^2} + \frac{3px^{p-2}((p+1)x^p + (p-1)y^p)f'(x)}{(x^p - y^p)^3}\\
&\quad - \frac{px^{p-3}((p+1)(p+2)x^{2p} + 4(p^2-1)x^p y^p + (p-1)(p-2)y^{2p})(f(x) - f(y))}{(x^p - y^p)^4},\\
\frac{\partial^4 B_f}{\partial x^4}(x,y) &= \frac{f^{(4)}(x)}{x^p - y^p} - \frac{4px^{p-1}f^{(3)}(x)}{(x^p - y^p)^2} + \frac{6px^{p-2}((p+1)x^p + (p-1)y^p)f''(x)}{(x^p - y^p)^3}\\
&\quad - \frac{4px^{p-3}((p+1)(p+2)x^{2p} + 4(p^2-1)x^p y^p + (p-1)(p-2)y^{2p})f'(x)}{(x^p - y^p)^4}\\
&\quad + \frac{px^{p-4}((p+1)(p+2)(p+3)x^{3p} + (p-1)(p-2)(p-3)y^{3p})(f(x) - f(y))}{(x^p - y^p)^5}\\
&\quad + \frac{px^{p-4}((p^2-1)x^p y^p((11p+8)x^p + (11p-8)y^p))(f(x) - f(y))}{(x^p - y^p)^5},
\end{aligned}$$

for all $x, y \in I_0$, $x \neq y$. In order to get the form of $\frac{\partial^i B_h}{\partial x^i}$ for all $i \in \{1,2,3,4\}$ it is enough to replace f by h and p by r.

It is easy to verify that

$$\lim_{y\to x}\frac{\partial^3 B_f}{\partial x^3}(x,y)=\frac{x^3 f^{(4)}(x)-2(p-1)x^2 f^{(3)}(x)+(p^2-1)(xf''(x)+f'(x))}{4px^{p+2}},$$

$$\lim_{y\to x}\frac{\partial^4 B_f}{\partial x^4}(x,y)=\frac{6x^4 f^{(5)}(x)-15(p-1)x^3 f^{(4)}(x)+10(p^2-1)x^2 f^{(3)}(x)}{30px^{p+3}}-\frac{15(p^2-1)xf''(x)+(p^2-1)(p^2-19)f'(x)}{30px^{p+3}},$$

and $\lim_{y\to x}\frac{\partial B_f}{\partial x}(x,y)$, $\lim_{y\to x}\frac{\partial^2 B_f}{\partial x^2}(x,y)$ are given, respectively, by (4.7), (4.8). Replacing f by h and p by r in the above formulas we get $\lim_{y\to x}\frac{\partial^i B_h}{\partial x^i}(x,y)$ for all $i\in\{1,2,3,4\}$.

Let us note that

$$\lim_{y\to x}\frac{\partial B_f}{\partial x}(x,y)=\frac{1}{2}u'(x), \tag{4.14}$$

$$\lim_{y\to x}\frac{\partial^2 B_f}{\partial x^2}(x,y)=\frac{1}{3}\left(u''(x)+\frac{p-1}{2x}u'(x)\right), \tag{4.15}$$

$$\lim_{y\to x}\frac{\partial^3 B_f}{\partial x^3}(x,y)=\frac{1}{4}\left(u^{(3)}(x)+\frac{p-1}{x}u''(x)-\frac{p-1}{x^2}u'(x)\right), \tag{4.16}$$

$$\begin{aligned}\lim_{y\to x}\frac{\partial^4 B_f}{\partial x^4}(x,y)&=\frac{1}{5}u^{(4)}(x)+\frac{3(p-1)}{10x}u^{(3)}(x)+\left(\frac{8(p-1)(p-2)}{15x^2}-\frac{(p-1)^2}{2x^2}\right)u''(x)\\&\quad+\left(\frac{(p-1)^3}{2x^3}-\frac{5(p-1)^2(p-2)}{6x^3}+\frac{3(p-1)(p-2)(p-3)}{10x^3}\right)u'(x).\end{aligned} \tag{4.17}$$

Replacing f, p and u, respectively, by h, r and v, in (4.14)–(4.17), we obtain $\lim_{y\to x}\frac{\partial^i B_h}{\partial x^i}$ for $i\in\{1,2,3,4\}$.

Letting $y\to x$ in Eq. (4.13) and applying (4.14)–(4.17) we get

$$\begin{aligned}&x^3\left[33\left(\frac{u^{(4)}}{u'}+\frac{v^{(4)}}{v'}\right)+55\left(\left(\frac{u''}{u'}\right)^3+\left(\frac{v''}{v'}\right)^3\right)-90\left(\frac{u''}{u'}\frac{u^{(3)}}{u'}+\frac{v''}{v'}\frac{v^{(3)}}{v'}\right)\right]\\&\quad+x^2\left[12\left((p+4)\frac{u^{(3)}}{u'}+(r+4)\frac{v^{(3)}}{v'}\right)-20\left((p+2)\left(\frac{u''}{u'}\right)^2+(r+2)\left(\frac{v''}{v'}\right)^2\right)+10\frac{u''}{u'}\frac{v''}{v'}\right]\\&\quad+4x\left[\left(5r+4p-3p^2-6\right)\frac{u''}{u'}+\left(5p+4r-3r^2-6\right)\frac{v''}{v'}\right]\\&\quad-8\left(p^3+r^3-p^2-r^2+p+r-5pr+3\right)=0,\end{aligned} \tag{4.18}$$

where $u^{(i)}$ and $v^{(i)}$ stand for $u^{(i)}(x)$ and $v^{(i)}(x)$ for all $i\in\{1,2,3,4\}$ and $x\in I_0$.

Since, by Lemma 1, for some $c\neq 0$,

$$u'(x)v'(x)=\frac{c}{x^{2(p+r+1)}},\quad x\in I_0,$$

for all $x\in I_0$ we have

$$\frac{v''(x)}{v'(x)}=-\frac{u''(x)}{u'(x)}-\frac{2(p+r+1)}{x},$$

$$\frac{v^{(3)}(x)}{v'(x)}=-\frac{u^{(3)}(x)}{u'(x)}+2\left(\frac{u''(x)}{u'(x)}\right)^2+\frac{4(p+r+1)}{x}\frac{u''(x)}{u'(x)}+\frac{2(p+r+1)(2p+2r+3)}{x^2},$$

$$\begin{aligned}\frac{v^{(4)}(x)}{v'(x)}&=-\frac{u^{(4)}(x)}{u'(x)}+6\frac{u''(x)}{u'(x)}\frac{u^{(3)}(x)}{u'(x)}+\frac{6(p+r+1)}{x}\frac{u^{(3)}(x)}{u'(x)}-\frac{12(p+r+1)}{x^3}\left(\frac{u''(x)}{u'(x)}\right)^2\\&\quad-6\left(\frac{u''(x)}{u'(x)}\right)^3-\frac{6(p+r+1)(2p+2r+3)}{x^2}\frac{u''(x)}{u'(x)}-\frac{4(p+r+1)(p+r+2)(2p+2r+3)}{x^3}.\end{aligned}$$

Applying above relations we can write Eq. (4.18) in the form

$$9x^3\left(\frac{u''(x)}{u'(x)}\frac{u^{(3)}(x)}{u'(x)}-\left(\frac{u''(x)}{u'(x)}\right)^3\right)-(13p+r)x^2\left(\frac{u''(x)}{u'(x)}\right)^2+3(5p+r+3)x^2\frac{u^{(3)}(x)}{u'(x)}$$
$$+\big((p+r)\big(2(p+r)+4p+7\big)+12p+3\big)x\frac{u''(x)}{u'(x)}+2(p+r)(p+r+1)(2p+1)=0. \tag{4.19}$$

Define the function $w_f : I_0 \to \mathbb{R}$ by

$$w_f(x) := \frac{u''(x)}{u'(x)}, \quad x \in I_0. \tag{4.20}$$

The function w_f is continuously differentiable in I_0 and

$$w'_f(x) = \frac{u^{(3)}(x)}{u'(x)} - \left(\frac{u''(x)}{u'(x)}\right)^2, \quad x \in I_0,$$

whence

$$\frac{u^{(3)}(x)}{u'(x)} = w'_f(x) + \big(w_f(x)\big)^2, \quad x \in I_0. \tag{4.21}$$

One can easily verify that

$$w_f(x) := \frac{x^2 f^{(3)}(x) - 2(p-1)xf''(x) + p(p-1)f'(x)}{x(xf''(x) - (p-1)f'(x))}, \quad x \in I_0.$$

Now from (4.20), (4.21) and (4.19) we obtain

$$2(p+r)(p+r+1)(2p+1) + 9x^3 w_f(x)w'_f(x) + \big(2(p+r)+9\big)x^2\big(w_f(x)\big)^2$$
$$+3(5p+r+3)x^2 w'_f(x) + \big[(p+r)\big(2(p+r)+4p+7\big)+12p+9\big]xw_f(x) = 0,$$

for all $x \in I_0$, which completes the proof of part (i). Part (ii) can be proved similarly. □

5. Main result

We begin this section with the following auxiliary

Lemma 3. *Let $I \subset (0,\infty)$ be an interval and let $p \neq 0$. The function $w : I \to \mathbb{R}$ of the class C^1 satisfies the differential equation*

$$\big(xw'(x) + w(x)\big)\big(3xw(x) + 4p + 3\big) = 0, \quad x \in I,$$

if, and only if, there exists $c \in \mathbb{R}$ such that

$$w(x) = \frac{c}{x}, \quad x \in I.$$

Since the proof of this lemma is elementary we omit it.

Theorem 2. *Let the functions $g : I \to \mathbb{R}$ and $k : I \to \mathbb{R}$ be given by*

$$g(x) = x^p, \qquad k(x) = x^{-p}, \quad x \in I,$$

for some $p \in \mathbb{R}\backslash\{0\}$. Suppose that $f : I \to \mathbb{R}$ and $h : I \to \mathbb{R}$ are differentiable functions such that $\frac{f'}{g'}$ and $\frac{h'}{k'}$ are injective in I.

The functions f, g, h, k satisfy Eq. (3.1) *if, and only if, either there exists $q \in \mathbb{R}\backslash\{0, p\}$ such that for all $x \in I$*

$$f(x) = A_1x^q + B_1x^p + C_1, \qquad h(x) = A_2x^{-q} + B_2x^{-p} + C_2, \tag{5.1}$$

or there exists $q \in \{0, p\}$ such that for all $x \in I$

$$f(x) = A_1x^q \ln x + B_1x^p + C_1, \qquad h(x) = A_2x^{-q}\ln x + B_2x^{-p} + C_2, \tag{5.2}$$

for some $A_i, B_i, C_i \in \mathbb{R}$, $A_i \neq 0$, $i \in \{1, 2\}$.

Proof. Assume that f, g, h, k satisfy Eq. (3.1). By Lemma 2 the functions f and h are of the class C^5 in an open nontrivial interval $I_0 \subset I$ and function $w_f : I_0 \to \mathbb{R}$ given by (4.11) satisfies Eq. (4.12), which in this case, when $r = -p$, can be written in the form

$$\big(xw'_f(x) + w_f(x)\big)\big(3xw_f(x) + 4p + 3\big) = 0, \quad x \in I_0.$$

Now, by Lemma 3, there exists $c \in \mathbb{R}$ such that

$$w_f(x) = \frac{c}{x}, \quad x \in I_0,$$

which, by (4.20), is equivalent to

$$\frac{u''(x)}{u'(x)} = \frac{c}{x}, \quad x \in I_0,$$

whence, for some $b \neq 0$,

$$u'(x) = bx^c, \quad x \in I_0. \tag{5.3}$$

Assume first that $c = -1$. In this case, by (5.3),

$$u'(x) = bx^{-1}, \quad x \in I_0, \tag{5.4}$$

for some $b \neq 0$ and, consequently, for some $d \in \mathbb{R}$,

$$u(x) = b \ln x + d, \quad x \in I_0.$$

Now applying the definition of u (see (4.1)) after simple calculation, we obtain

$$f(x) = A_1 x^p \ln x + B_1 x^p + C_1, \quad x \in I_0, \tag{5.5}$$

for some $A_1 \neq 0$ and $B_1, C_1 \in \mathbb{R}$. By Lemma 1 there exists $a \neq 0$ such that

$$u'(x)v'(x) = \frac{a}{x^2}, \quad x \in I_0,$$

that is, by (5.4),

$$v'(x) = \frac{a}{b}\frac{1}{x}, \quad x \in I_0,$$

whence, by the definition of v (see (4.1)), we obtain

$$h(x) = A_2 x^{-p} \ln x + B_2 x^{-p} + C_2, \quad x \in I_0, \tag{5.6}$$

for some $A_2 \neq 0$ and $B_2, C_2 \in \mathbb{R}$.

In the case when $c \neq -1$, by (5.3), we obtain

$$u(x) = \frac{b}{c+1}x^{c+1} + d, \quad x \in I_0,$$

for some $b \neq 0$ and $d \in \mathbb{R}$. By the definition of u (see (4.1)), we hence get

$$f'(x) = \frac{bp}{c+1}x^{p+c} + dpx^{p-1}, \quad x \in I_0. \tag{5.7}$$

If $p + c = -1$ then

$$f(x) = -b \ln x + dx^p + C_1, \quad x \in I_0,$$

for some $b \neq 0$ and $d, C_1 \in \mathbb{R}$. Thus

$$f(x) = A_1 \ln x + B_1 x^p + C_1, \quad x \in I_0, \tag{5.8}$$

for some $A_1 \neq 0$ and $B_1, C_1 \in \mathbb{R}$.

Now repeating reasoning as in previous part one can easily show that in this case

$$h(x) = A_2 \ln x + B_2 x^{-p} + C_2, \quad x \in I_0, \tag{5.9}$$

for some $A_2 \neq 0$ and $B_2, C_2 \in \mathbb{R}$.

If $p+c \neq -1$ from (5.7) we get

$$f(x) = \frac{bp}{(c+1)(p+c+1)} x^{p+c+1} + dx^p + C_1, \quad x \in I_0,$$

where $p+c+1 \neq 0$. Since $c \neq -1$, we have $p+c+1 \neq p$. Putting $q := p+c+1$, $A_1 := \frac{bp}{(c+1)(p+c+1)}$ and $B_1 := d$ we hence get

$$f(x) = A_1 x^q + B_1 x^p + C_1, \quad x \in I_0,$$

where $q \in \mathbb{R} \setminus \{0, p\}$, $A_1 \neq 0$, $B_1 \in \mathbb{R}$.

Applying again Lemma 1 and (4.1) we obtain

$$h(x) = A_2 x^{-q} + B_2 x^{-p} + C_2, \quad x \in I_0,$$

for some $A_2 \neq 0$ and $B_2, C_2 \in \mathbb{R}$.

If $p+c = -1$, then $q = 0$ and, in the case when $c = -1$, we have $q = p$. Applying formulas (5.5) and (5.8), for $q \in \{0, p\}$ we get

$$f(x) = A_1 x^q \ln x + B_1 x^p + C_1, \quad x \in I_0,$$

and similarly for $q \in \{0, p\}$, applying formulas (5.6) and (5.9), we obtain

$$h(x) = A_2 x^{-q} \ln x + B_2 x^{-p} + C_2, \quad x \in I_0.$$

Since f and h are of the class C^∞ in $(0, +\infty)$ we infer that $I_0 = I$. Thus we have proved the "only if" part of our theorem.

To prove converse implication assume that f and h are given by (5.1) in I. Then, it is easy to verify that

$$\left(\frac{f'}{g'}\right)^{-1} \left(\frac{f(x) - f(y)}{g(x) - g(y)}\right) = \left(\frac{p(x^q - y^q)}{q(x^p - y^p)}\right)^{\frac{1}{q-p}}, \quad x, y \in I,\ x \neq y,$$

and

$$\left(\frac{h'}{k'}\right)^{-1} \left(\frac{h(x) - h(y)}{k(x) - k(y)}\right) = \left(\frac{p(x^{-q} - y^{-q})}{q(x^{-p} - y^{-p})}\right)^{\frac{-1}{q-p}}, \quad x, y \in I,\ x \neq y,$$

where $pq \neq 0$, $p \neq q$. Hence

$$\left(\frac{p(x^q - y^q)}{q(x^p - y^p)}\right)^{\frac{1}{q-p}} \left(\frac{p(x^{-q} - y^{-q})}{q(x^{-p} - y^{-p})}\right)^{\frac{-1}{q-p}} = xy, \quad x, y \in I,\ x \neq y,$$

which means that the functions f, g, h and k satisfy Eq. (3.1). We omit similar calculations in the remaining cases. The proof is completed. □

Corollary 1. *Let the functions $g : I \to \mathbb{R}$ and $k : I \to \mathbb{R}$ be given by*

$$g(x) = x^p, \qquad k(x) = x^{-p}, \quad x \in I,$$

for some $p \in \mathbb{R} \setminus \{0\}$. Suppose that $f : I \to \mathbb{R}$ and $h : I \to \mathbb{R}$ are differentiable functions such that $\frac{f'}{g'}$ and $\frac{h'}{k'}$ are injective in I.

The geometric mean G is $(D^{f,g}, D^{h,k})$-invariant if, and only if, there exists $q \in \mathbb{R}$ such that

$$D^{f,g} = E_{p,q} \quad \text{and} \quad D^{h,k} = E_{-p,-q},$$

where $E_{p,q} : I^2 \to I$ is given by

$$E_{p,q}(x, y) = \begin{cases} \left(\frac{p(x^q - y^q)}{q(x^p - y^p)}\right)^{\frac{1}{q-p}}, & pq \neq 0,\ p \neq q, \\ \left(\frac{x^p - y^p}{p(\ln x - \ln y)}\right)^{\frac{1}{p}}, & p \neq 0,\ q = 0, \\ e^{-\frac{1}{p}} \left(\frac{x^{x^p}}{y^{y^p}}\right)^{\frac{1}{x^p - y^p}}, & p = q \neq 0, \end{cases}$$

for all $x, y \in I$, $x \neq y$, and $E_{p,q}(x, x) = x$, $x \in I$.

Proof. First assume that the geometric mean G is $(D^{f,g}, D^{h,k})$-invariant, that is the functions f, g, h and k satisfy Eq. (3.1). By Theorem 2 the functions f and h are either given by (5.1) or by (5.2). Hence, by the definition of the Cauchy mean, we obtain that either $q \in \mathbb{R} \setminus \{0, p\}$ and

$$D^{f,g}(x, y) = \left(\frac{p(x^q - y^q)}{q(x^p - y^p)}\right)^{\frac{1}{q-p}}, \qquad D^{h,k}(x, y) = \left(\frac{p(x^{-q} - y^{-q})}{q(x^{-p} - y^{-p})}\right)^{\frac{1}{p-q}},$$

or $q = 0$ and

$$D^{f,g}(x, y) = \left(\frac{x^p - y^p}{p(\ln x - \ln y)}\right)^{\frac{1}{p}}, \qquad D^{h,k}(x, y) = \left(\frac{x^{-p} - y^{-p}}{-p(\ln x - \ln y)}\right)^{\frac{1}{-p}},$$

or $q = p$ and

$$D^{f,g}(x, y) = e^{-\frac{1}{p}} \left(\frac{x^{x^p}}{y^{y^p}}\right)^{\frac{1}{x^p - y^p}}, \qquad D^{h,k}(x, y) = e^{\frac{1}{p}} \left(\frac{x^{x^{-p}}}{y^{y^{-p}}}\right)^{\frac{1}{x^{-p} - y^{-p}}},$$

for all $x, y \in I$, $x \neq y$. This completes the proof of this part.

The proof of the converse implication is given in [9]. $\square$

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